

Sharp estimates for the gradient of the generalized Poisson integral for a half-space

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Dedicated to Vakhtang Kokilashvili on the occasion of his 80th birthday

Abstract. A representation of the sharp coefficient in a pointwise estimate for the gradient of the generalized Poisson integral of a function f on \mathbb{R}^n is obtained under the assumption that f belongs to L^p . The explicit value of the coefficient is found for the cases $p = 1$ and $p = 2$.

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1 Introduction

In the paper [3] (see also [6]) a representation for the sharp coefficient $\mathcal{K}_p(x)$ in the inequality

$$|\nabla u(x)| \leq \mathcal{K}_p(x) \|u\|_p$$

was found, where u is harmonic function in the half-space $\mathbb{R}_+^{n+1} = \{x = (x', x_{n+1}) : x' \in \mathbb{R}^n, x_{n+1} > 0\}$, represented by the Poisson integral with boundary values in $L^p(\mathbb{R}^n)$, $\|\cdot\|_p$ is the norm in $L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, $x \in \mathbb{R}_+^{n+1}$. It was shown that

$$\mathcal{K}_p(x) = \frac{K_p}{x_{n+1}^{(n+p)/p}}$$

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and explicit formulas for K_1 and K_2 were given. Namely,

$$K_1 = \frac{2n}{\omega_{n+1}}, \quad K_2 = \sqrt{\frac{n(n+1)}{2^{n+1}\omega_{n+1}}},$$

where $\omega_n = 2\pi^{n/2}/\Gamma(n/2)$ is the area of the unit sphere in \mathbb{R}^n .

In [3] it was shown that the sharp coefficients in pointwise estimates for the absolute value of the normal derivative and the modulus of the gradient of a harmonic function in the half-space coincide for the case $p = 1$ as well as for the case $p = 2$.

Similar results for the gradient and the radial derivative of a harmonic function in the multidimensional ball with boundary values from L^p for $p = 1, 2$ in [4] were obtained.

Thus, the L^1, L^2 -analogues of Khavinson's problem [1] were solved in [3, 4] for harmonic functions in the multidimensional half-space and the ball.

We note that explicit sharp coefficients in the inequality for the first derivative of analytic function in the half-plane and the disk with boundary values of the real-part from L^p in [2, 5, 7] were found.

In this paper we treat a generalization of the problem considered in our work [3]. Here we consider the generalized Poisson integral

$$u_f(x) = k_{n,\alpha} \int_{\mathbb{R}^n} \frac{x_{n+1}^\alpha}{|y-x|^{n+\alpha}} f(y') dy'$$

with $f \in L^p(\mathbb{R}^n)$, $\alpha > -(n/p)$, $1 \leq p \leq \infty$, where $x \in \mathbb{R}_+^{n+1}$, $y = (y', 0)$, $y' \in \mathbb{R}^n$, and $k_{n,\alpha}$ is a normalization constant. In the case $\alpha = 1$ the last integral coincides with the Poisson integral for a half-space.

In Section 2 we obtain a representation for the sharp coefficient $\mathcal{C}_p(x)$ in the inequality

$$|\nabla u_f(x)| \leq \mathcal{C}_p(x) \|f\|_p,$$

where

$$\mathcal{C}_p(x) = \frac{C_p}{x_{n+1}^{(n+p)/p}}$$

and the constant C_p is characterized in terms of an extremal problem on the unit sphere \mathbb{S}^n in \mathbb{R}^{n+1} .

In Section 3 we reduce this extremal problem to that of finding of the supremum of a certain double integral, depending on a scalar parameter and show that

$$C_1 = k_{n,\alpha} n$$

if $-n < \alpha \leq n$, and

$$C_2 = \sqrt{\omega_{n-1}} k_{n,\alpha} \left\{ \frac{\sqrt{\pi}(n+\alpha)n(n+2)\Gamma(\frac{n}{2}-1)\Gamma(\frac{n}{2}+\alpha)}{8(n+1+\alpha)\Gamma(n+\alpha)} \right\}^{1/2}$$

if $-(n/2) < \alpha \leq n(n+1)/2$.

It is shown that the sharp coefficients in pointwise estimates for the absolute value of the normal derivative and the modulus of the gradient of the generalized Poisson integral for a half-space coincide in the case $p = 1$ as well as in the case $p = 2$.

2 Representation for the sharp constant in inequality for the gradient in terms of an extremal problem on the unit sphere

We introduce some notation used henceforth. Let $\mathbb{R}_+^{n+1} = \{x = (x', x_{n+1}) : x' = (x_1, \dots, x_n) \in \mathbb{R}^n, x_{n+1} > 0\}$, $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$, $\mathbb{S}_+^n = \{x \in \mathbb{R}^{n+1} : |x| = 1, x_{n+1} > 0\}$ and $\mathbb{S}_-^n = \{x \in \mathbb{R}^{n+1} : |x| = 1, x_{n+1} < 0\}$. Let e_σ stand for the $n+1$ -dimensional unit vector joining the origin to a point σ on the sphere \mathbb{S}^n .

By $\|\cdot\|_p$ we denote the norm in the space $L^p(\mathbb{R}^n)$, that is

$$\|f\|_p = \left\{ \int_{\mathbb{R}^n} |f(x')|^p dx' \right\}^{1/p},$$

if $1 \leq p < \infty$, and $\|f\|_\infty = \text{ess sup}\{|f(x')| : x' \in \mathbb{R}^n\}$.

Let the function in \mathbb{R}_+^{n+1} be represented as the generalized Poisson integral

$$u_f(x) = k_{n,\alpha} \int_{\mathbb{R}^n} \frac{x_{n+1}^\alpha}{|y-x|^{n+\alpha}} f(y') dy' \quad (2.1)$$

with $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, where $y = (y', 0)$, $y' \in \mathbb{R}^n$,

$$k_{n,\alpha} = \left\{ \int_{\mathbb{R}^n} \frac{x_{n+1}^\alpha}{|y-x|^{n+\alpha}} dy' \right\}^{-1} = \frac{\Gamma\left(\frac{n+\alpha}{2}\right)}{\pi^{n/2} \Gamma\left(\frac{\alpha}{2}\right)}, \quad (2.2)$$

and

$$\alpha > -\frac{n}{p}. \quad (2.3)$$

Now, we find a representation for the best coefficient $\mathcal{C}_p(x; \mathbf{z})$ in the inequality for the absolute value of derivative of $u_f(x)$ in an arbitrary direction $\mathbf{z} \in \mathbb{S}^n$, $x \in \mathbb{R}_+^{n+1}$. In particular, we obtain a formula for the constant in a similar inequality for the modulus of the gradient.

Proposition 1. *Let x be an arbitrary point in \mathbb{R}_+^{n+1} and let $\mathbf{z} \in \mathbb{S}^n$. The sharp coefficient $\mathcal{C}_p(x; \mathbf{z})$ in the inequality*

$$|(\nabla u_f(x), \mathbf{z})| \leq \mathcal{C}_p(x; \mathbf{z}) \|f\|_p$$

is given by

$$\mathcal{C}_p(x; \mathbf{z}) = \frac{C_p(\mathbf{z})}{x_{n+1}^{(n+p)/p}}, \quad (2.4)$$

where

$$C_1(\mathbf{z}) = k_{n,\alpha} \sup_{\sigma \in \mathbb{S}_+^n} |(\alpha e_{n+1} - (n+\alpha)(e_\sigma, e_{n+1})e_\sigma, \mathbf{z})| (e_\sigma, e_{n+1})^{n+\alpha}, \quad (2.5)$$

$$C_p(\mathbf{z}) = k_{n,\alpha} \left\{ \int_{\mathbb{S}_+^n} |(\alpha e_{n+1} - (n+\alpha)(e_\sigma, e_{n+1})e_\sigma, \mathbf{z})|^{\frac{p}{p-1}} (e_\sigma, e_{n+1})^{\frac{(\alpha-1)p+n+1}{p-1}} d\sigma \right\}^{\frac{p-1}{p}} \quad (2.6)$$

for $1 < p < \infty$, and

$$C_\infty(\mathbf{z}) = k_{n,\alpha} \int_{\mathbb{S}_+^n} |(\alpha e_{n+1} - (n+\alpha)(e_\sigma, e_{n+1})e_\sigma, \mathbf{z})| (e_\sigma, e_{n+1})^{\alpha-1} d\sigma. \quad (2.7)$$

In particular, the sharp coefficient $\mathcal{C}_p(x)$ in the inequality

$$|\nabla u_f(x)| \leq \mathcal{C}_p(x) \|f\|_p$$

is given by

$$\mathcal{C}_p(x) = \frac{C_p}{x_{n+1}^{(n+p)/p}}, \quad (2.8)$$

where

$$C_p = \sup_{|z|=1} C_p(z). \quad (2.9)$$

Proof. Let $x = (x', x_{n+1})$ be a fixed point in \mathbb{R}_+^{n+1} . The representation (2.1) implies

$$\frac{\partial u_f}{\partial x_i} = k_{n,\alpha} \int_{\mathbb{R}^n} \left[\frac{\delta_{ni} \alpha x_{n+1}^{\alpha-1}}{|y-x|^{n+\alpha}} + \frac{(n+\alpha) x_{n+1}^\alpha (y_i - x_i)}{|y-x|^{n+2+\alpha}} \right] f(y') dy',$$

that is

$$\begin{aligned} \nabla u_f(x) &= k_{n,\alpha} x_{n+1}^{\alpha-1} \int_{\mathbb{R}^n} \left[\frac{\alpha \mathbf{e}_{n+1}}{|y-x|^{n+\alpha}} + \frac{(n+\alpha) x_{n+1} (y-x)}{|y-x|^{n+2+\alpha}} \right] f(y') dy' \\ &= k_{n,\alpha} x_{n+1}^{\alpha-1} \int_{\mathbb{R}^n} \frac{\alpha \mathbf{e}_{n+1} - (n+\alpha) (\mathbf{e}_{xy}, \mathbf{e}_{n+1}) \mathbf{e}_{xy}}{|y-x|^{n+\alpha}} f(y') dy', \end{aligned}$$

where $\mathbf{e}_{xy} = (y-x)|y-x|^{-1}$. For any $\mathbf{z} \in \mathbb{S}^n$,

$$(\nabla u_f(x), \mathbf{z}) = k_{n,\alpha} x_{n+1}^{\alpha-1} \int_{\mathbb{R}^n} \frac{(\alpha \mathbf{e}_{n+1} - (n+\alpha) (\mathbf{e}_{xy}, \mathbf{e}_{n+1}) \mathbf{e}_{xy}, \mathbf{z})}{|y-x|^{n+\alpha}} f(y') dy'. \quad (2.10)$$

Hence,

$$\mathcal{C}_1(x; \mathbf{z}) = k_{n,\alpha} x_{n+1}^{\alpha-1} \sup_{y \in \partial \mathbb{R}^n} \frac{|(\alpha \mathbf{e}_{n+1} - (n+\alpha) (\mathbf{e}_{xy}, \mathbf{e}_{n+1}) \mathbf{e}_{xy}, \mathbf{z})|}{|y-x|^{n+\alpha}}, \quad (2.11)$$

and

$$\mathcal{C}_p(x; \mathbf{z}) = k_{n,\alpha} x_{n+1}^{\alpha-1} \left\{ \int_{\mathbb{R}^n} \frac{|(\alpha \mathbf{e}_{n+1} - (n+\alpha) (\mathbf{e}_{xy}, \mathbf{e}_{n+1}) \mathbf{e}_{xy}, \mathbf{z})|^q}{|y-x|^{(n+\alpha)q}} dy' \right\}^{1/q} \quad (2.12)$$

for $1 < p \leq \infty$, where $p^{-1} + q^{-1} = 1$.

Taking into account the equality

$$\frac{x_{n+1}}{|y-x|} = (\mathbf{e}_{xy}, -\mathbf{e}_{n+1}), \quad (2.13)$$

by (2.11) we obtain

$$\begin{aligned} \mathcal{C}_1(x; \mathbf{z}) &= k_{n,\alpha} x_{n+1}^{\alpha-1} \sup_{y \in \partial \mathbb{R}^n} \frac{|(\alpha \mathbf{e}_{n+1} - (n+\alpha) (\mathbf{e}_{xy}, \mathbf{e}_{n+1}) \mathbf{e}_{xy}, \mathbf{z})|}{x_{n+1}^{n+\alpha}} \left(\frac{x_{n+1}}{|y-x|} \right)^{n+\alpha} \\ &= \frac{k_{n,\alpha}}{x_{n+1}^{n+1}} \sup_{\sigma \in \mathbb{S}_-^n} |(\alpha \mathbf{e}_{n+1} - (n+\alpha) (\mathbf{e}_\sigma, \mathbf{e}_{n+1}) \mathbf{e}_\sigma, \mathbf{z})| (\mathbf{e}_\sigma, -\mathbf{e}_{n+1})^{n+\alpha}. \end{aligned}$$

Replacing here \mathbf{e}_σ by $-\mathbf{e}_\sigma$, we arrive at (2.4) for $p = 1$ with the sharp constant (2.5).

Let $1 < p \leq \infty$. Using (2.13) and the equality

$$\frac{1}{|y-x|^{(n+\alpha)q}} = \frac{1}{x_{n+1}^{(n+\alpha)q-n}} \left(\frac{x_{n+1}}{|y-x|} \right)^{(n+\alpha)q-n-1} \frac{x_{n+1}}{|y-x|^{n+1}},$$

and replacing q by $p/(p-1)$ in (2.12), we conclude that (2.4) holds with the sharp constant

$$C_p(\mathbf{z}) = k_{n,\alpha} \left\{ \int_{\mathbb{S}_-^n} |(\alpha \mathbf{e}_{n+1} - (n+\alpha)(\mathbf{e}_\sigma, \mathbf{e}_{n+1})\mathbf{e}_\sigma, \mathbf{z})|^{\frac{p}{p-1}} (\mathbf{e}_\sigma, -\mathbf{e}_{n+1})^{\frac{(\alpha-1)p+n+1}{p-1}} d\sigma \right\}^{\frac{p-1}{p}},$$

where $\mathbb{S}_-^n = \{\sigma \in \mathbb{S}^n : (\mathbf{e}_\sigma, \mathbf{e}_{n+1}) < 0\}$. Replacing here \mathbf{e}_σ by $-\mathbf{e}_\sigma$, we arrive at (2.6) for $1 < p < \infty$ and at (2.7) for $p = \infty$.

By (2.10) we have

$$|\nabla u_f(x)| = k_{n,\alpha} x_{n+1}^{\alpha-1} \sup_{|\mathbf{z}|=1} \int_{\mathbb{R}^n} \frac{(\alpha \mathbf{e}_{n+1} - (n+\alpha)(\mathbf{e}_{xy}, \mathbf{e}_{n+1})\mathbf{e}_{xy}, \mathbf{z})}{|y-x|^{n+\alpha}} f(y') dy'.$$

Hence, by the permutation of suprema, (2.12), (2.11) and (2.4),

$$\begin{aligned} \mathcal{C}_p(x) &= k_{n,\alpha} x_{n+1}^{\alpha-1} \sup_{|\mathbf{z}|=1} \left\{ \int_{\mathbb{R}^{n+1}} \frac{|(\alpha \mathbf{e}_{n+1} - (n+\alpha)(\mathbf{e}_{xy}, \mathbf{e}_{n+1})\mathbf{e}_{xy}, \mathbf{z})|^q}{|y-x|^{(n+\alpha)q}} dy' \right\}^{1/q} \\ &= \sup_{|\mathbf{z}|=1} \mathcal{C}_p(x; \mathbf{z}) = \sup_{|\mathbf{z}|=1} C_p(\mathbf{z}) x_{n+1}^{-(n+p)/p} \end{aligned} \quad (2.14)$$

for $1 < p \leq \infty$, and

$$\begin{aligned} \mathcal{C}_1(x) &= k_{n,\alpha} x_{n+1}^{\alpha-1} \sup_{|\mathbf{z}|=1} \sup_{y \in \partial \mathbb{R}^n} \frac{|(\alpha \mathbf{e}_{n+1} - (n+\alpha)(\mathbf{e}_{xy}, \mathbf{e}_{n+1})\mathbf{e}_{xy}, \mathbf{z})|}{|y-x|^{n+\alpha}} \\ &= \sup_{|\mathbf{z}|=1} \mathcal{C}_1(x; \mathbf{z}) = \sup_{|\mathbf{z}|=1} C_1(\mathbf{z}) x_{n+1}^{-(n+1)}. \end{aligned} \quad (2.15)$$

Using the notation (2.9) in (2.14) and (2.15), we arrive at (2.8). \square

Remark. Formula (2.6) for the coefficient $C_p(\mathbf{z})$, $1 < p < \infty$, can be written with the integral over the whole sphere \mathbb{S}^n in \mathbb{R}^{n+1} ,

$$C_p(\mathbf{z}) = \frac{k_{n,\alpha}}{2^{(p-1)/p}} \left\{ \int_{\mathbb{S}^n} |(\alpha \mathbf{e}_{n+1} - (n+\alpha)(\mathbf{e}_\sigma, \mathbf{e}_{n+1})\mathbf{e}_\sigma, \mathbf{z})|^{\frac{p}{p-1}} (\mathbf{e}_\sigma, \mathbf{e}_{n+1})^{\frac{(\alpha-1)p+n+1}{p-1}} d\sigma \right\}^{\frac{p-1}{p}}$$

A similar remark relates (2.7):

$$C_\infty(\mathbf{z}) = \frac{k_{n,\alpha}}{2} \int_{\mathbb{S}^n} |(\alpha \mathbf{e}_{n+1} - (n+\alpha)(\mathbf{e}_\sigma, \mathbf{e}_{n+1})\mathbf{e}_\sigma, \mathbf{z})| |(\mathbf{e}_\sigma, \mathbf{e}_{n+1})|^{\alpha-1} d\sigma, \quad (2.16)$$

as well as formula (2.5):

$$C_1(\mathbf{z}) = k_{n,\alpha} \sup_{\sigma \in \mathbb{S}^n} |(\alpha \mathbf{e}_{n+1} - (n+\alpha)(\mathbf{e}_\sigma, \mathbf{e}_{n+1})\mathbf{e}_\sigma, \mathbf{z})| |(\mathbf{e}_\sigma, \mathbf{e}_{n+1})|^{n+\alpha}.$$

3 Reduction of the extremal problem to finding of the supremum by parameter of a double integral. The cases $p = 1$ and $p = 2$

The next assertion is based on the representation for C_p , obtained in Proposition 1.

Proposition 2. *Let $f \in L^p(\mathbb{R}^n)$, and let x be an arbitrary point in \mathbb{R}_+^{n+1} . The sharp coefficient $C_p(x)$ in the inequality*

$$|\nabla u_f(x)| \leq C_p(x) \|f\|_p \quad (3.1)$$

is given by

$$C_p(x) = \frac{C_p}{x_{n+1}^{(n+p)/p}}, \quad (3.2)$$

where

$$C_p = (\omega_{n-1})^{(p-1)/p} k_{n,\alpha} \sup_{\gamma \geq 0} \frac{1}{\sqrt{1+\gamma^2}} \left\{ \int_0^\pi d\varphi \int_0^{\pi/2} \mathcal{F}_{n,p}(\varphi, \vartheta; \gamma) d\vartheta \right\}^{\frac{p-1}{p}}, \quad (3.3)$$

if $1 < p < \infty$. Here

$$\mathcal{F}_{n,p}(\varphi, \vartheta; \gamma) = |\mathcal{G}_n(\varphi, \vartheta; \gamma)|^{p/(p-1)} \cos((\alpha-1)p+n+1)/(p-1) \vartheta \sin^{n-1} \vartheta \sin^{n-2} \varphi \quad (3.4)$$

with

$$\mathcal{G}_n(\varphi, \vartheta; \gamma) = ((n+\alpha) \cos^2 \vartheta - \alpha) + \gamma(n+\alpha) \cos \vartheta \sin \vartheta \cos \varphi. \quad (3.5)$$

In addition,

$$C_1 = k_{n,\alpha} n \quad (3.6)$$

if $-n < \alpha \leq n$.

In particular,

$$C_2 = \sqrt{\omega_{n-1}} k_{n,\alpha} \left\{ \frac{\sqrt{\pi}(n+\alpha)n(n+2)\Gamma(\frac{n}{2}-1)\Gamma(\frac{n}{2}+\alpha)}{8(n+1+\alpha)\Gamma(n+\alpha)} \right\}^{1/2}$$

for $-(n/2) < \alpha \leq n(n+1)/2$.

For $p = 1$ and $p = 2$ the coefficient $C_p(x)$ is sharp in conditions of the Proposition also in the weaker inequality obtained from (3.1) by replacing ∇u_f by $\partial u_f / \partial x_{n+1}$.

Proof. The equality (3.2) was proved in Proposition 1.

(i) Let $p = 1$. Using (2.5), (2.9) and the permutability of two suprema, we find

$$\begin{aligned} C_1 &= k_{n,\alpha} \sup_{|z|=1} \sup_{\sigma \in \mathbb{S}_+^n} |(\alpha e_{n+1} - (n+\alpha)(e_\sigma, e_{n+1})e_\sigma, z)| (e_\sigma, e_{n+1})^{n+\alpha} \\ &= k_{n,\alpha} \sup_{\sigma \in \mathbb{S}_+^n} |\alpha e_{n+1} - (n+\alpha)(e_\sigma, e_{n+1})e_\sigma| (e_\sigma, e_{n+1})^{n+\alpha}. \end{aligned} \quad (3.7)$$

Taking into account the equality

$$\begin{aligned}
& |\alpha \mathbf{e}_{n+1} - (n + \alpha)(\mathbf{e}_\sigma, \mathbf{e}_{n+1})\mathbf{e}_\sigma| \\
&= \left(\alpha \mathbf{e}_{n+1} - (n + \alpha)(\mathbf{e}_\sigma, \mathbf{e}_{n+1})\mathbf{e}_\sigma, \alpha \mathbf{e}_{n+1} - (n + \alpha)(\mathbf{e}_\sigma, \mathbf{e}_{n+1})\mathbf{e}_\sigma \right)^{1/2} \\
&= \left(\alpha^2 + ((n + \alpha)^2 - 2\alpha(n + \alpha))(\mathbf{e}_\sigma, \mathbf{e}_{n+1})^2 \right)^{1/2},
\end{aligned}$$

and using (2.3), (3.7), we arrive at the sharp constant (3.6) for $-n < \alpha \leq n$.

Furthermore, by (2.5),

$$C_1(\mathbf{e}_{n+1}) = k_{n,\alpha} \sup_{\sigma \in \mathbb{S}_+^n} |\alpha - (n + \alpha)(\mathbf{e}_\sigma, \mathbf{e}_{n+1})^2| (\mathbf{e}_\sigma, \mathbf{e}_{n+1})^{n+\alpha} \geq k_{n,\alpha} n.$$

Hence, by $C_1 \geq C_1(\mathbf{e}_{n+1})$ and by (3.6) we obtain $C_1 = C_1(\mathbf{e}_{n+1})$, which completes the proof in the case $p = 1$.

(ii) Let $1 < p < \infty$. Since the integrand in (2.6) does not change when $\mathbf{z} \in \mathbb{S}^n$ is replaced by $-\mathbf{z}$, we may assume that $z_{n+1} = (\mathbf{e}_{n+1}, \mathbf{z}) > 0$ in (2.9).

Let $\mathbf{z}' = \mathbf{z} - z_{n+1}\mathbf{e}_{n+1}$. Then $(\mathbf{z}', \mathbf{e}_{n+1}) = 0$ and hence $z_{n+1}^2 + |\mathbf{z}'|^2 = 1$. Analogously, with $\sigma = (\sigma_1, \dots, \sigma_n, \sigma_{n+1}) \in \mathbb{S}_+^n$, we associate the vector $\boldsymbol{\sigma}' = \mathbf{e}_\sigma - \sigma_{n+1}\mathbf{e}_{n+1}$.

Using the equalities $(\boldsymbol{\sigma}', \mathbf{e}_{n+1}) = 0$, $\sigma_{n+1} = \sqrt{1 - |\boldsymbol{\sigma}'|^2}$ and $(\mathbf{z}', \mathbf{e}_{n+1}) = 0$, we find an expression for $(\alpha \mathbf{e}_{n+1} - (n + \alpha)(\mathbf{e}_\sigma, \mathbf{e}_{n+1})\mathbf{e}_\sigma, \mathbf{z})$ as a function of $\boldsymbol{\sigma}'$:

$$\begin{aligned}
& (\alpha \mathbf{e}_{n+1} - (n + \alpha)(\mathbf{e}_\sigma, \mathbf{e}_{n+1})\mathbf{e}_\sigma, \mathbf{z}) = \alpha z_{n+1} - (n + \alpha)\sigma_{n+1}(\mathbf{e}_\sigma, \mathbf{z}) \\
&= \alpha z_{n+1} - (n + \alpha)\sigma_{n+1}(\boldsymbol{\sigma}' + \sigma_{n+1}\mathbf{e}_{n+1}, \mathbf{z}' + z_{n+1}\mathbf{e}_{n+1}) \\
&= \alpha z_{n+1} - (n + \alpha)\sigma_{n+1}[(\boldsymbol{\sigma}', \mathbf{z}') + z_{n+1}\sigma_{n+1}] \\
&= -[(n + \alpha)(1 - |\boldsymbol{\sigma}'|^2) - \alpha]z_{n+1} - (n + \alpha)\sqrt{1 - |\boldsymbol{\sigma}'|^2}(\boldsymbol{\sigma}', \mathbf{z}'). \tag{3.8}
\end{aligned}$$

Let $\mathbb{B}^n = \{x' = (x_1, \dots, x_n) \in \mathbb{R}^n : |x'| < 1\}$. By (2.6) and (3.8), taking into account that $d\sigma = d\sigma'/\sqrt{1 - |\boldsymbol{\sigma}'|^2}$, we may write (2.9) as

$$\begin{aligned}
C_p &= k_{n,\alpha} \sup_{\mathbf{z} \in \mathbb{S}_+^n} \left\{ \int_{\mathbb{B}^n} \frac{\mathcal{H}_{n,p}(|\boldsymbol{\sigma}'|, (\boldsymbol{\sigma}', \mathbf{z}'))(1 - |\boldsymbol{\sigma}'|^2)^{(\alpha p + n + 1)/2(p-1)}}{\sqrt{1 - |\boldsymbol{\sigma}'|^2}} d\sigma' \right\}^{\frac{p-1}{p}} \\
&= k_{n,\alpha} \sup_{\mathbf{z} \in \mathbb{S}_+^n} \left\{ \int_{\mathbb{B}^n} \mathcal{H}_{n,p}(|\boldsymbol{\sigma}'|, (\boldsymbol{\sigma}', \mathbf{z}'))(1 - |\boldsymbol{\sigma}'|^2)^{((\alpha-2)p + n + 2)/2(p-1)} d\sigma' \right\}^{\frac{p-1}{p}}, \tag{3.9}
\end{aligned}$$

where

$$\mathcal{H}_{n,p}(|\boldsymbol{\sigma}'|, (\boldsymbol{\sigma}', \mathbf{z}')) = \left| [(n + \alpha)(1 - |\boldsymbol{\sigma}'|^2) - \alpha]z_{n+1} + (n + \alpha)\sqrt{1 - |\boldsymbol{\sigma}'|^2}(\boldsymbol{\sigma}', \mathbf{z}') \right|^{p/(p-1)}. \tag{3.10}$$

Using the well known formula (see e.g. [8], **3.3.2(3)**),

$$\int_{B^n} g(|\mathbf{x}|, (\mathbf{a}, \mathbf{x})) d\mathbf{x} = \omega_{n-1} \int_0^1 r^{n-1} dr \int_0^\pi g(r, |\mathbf{a}|r \cos \varphi) \sin^{n-2} \varphi d\varphi,$$

we obtain

$$\begin{aligned} & \int_{\mathbb{B}^n} \mathcal{H}_{n,p}(|\boldsymbol{\sigma}'|, (\boldsymbol{\sigma}', \mathbf{z}')) (1 - |\boldsymbol{\sigma}'|^2)^{((\alpha-2)p+n+2)/2(p-1)} d\boldsymbol{\sigma}' \\ &= \omega_{n-1} \int_0^1 r^{n-1} (1-r^2)^{((\alpha-2)p+n+2)/2(p-1)} dr \int_0^\pi \mathcal{H}_{n,p}(r, r|\mathbf{z}'| \cos \varphi) \sin^{n-2} \varphi d\varphi. \end{aligned}$$

Making the change of variable $r = \sin \vartheta$ in the right-hand side of the last equality, we find

$$\begin{aligned} & \int_{\mathbb{B}^n} \mathcal{H}_{n,p}(|\boldsymbol{\sigma}'|, (\boldsymbol{\sigma}', \mathbf{z}')) (1 - |\boldsymbol{\sigma}'|^2)^{\frac{(\alpha-2)p+n+2}{2(p-1)}} d\boldsymbol{\sigma}' \tag{3.11} \\ &= \omega_{n-1} \int_0^\pi \sin^{n-2} \varphi d\varphi \int_0^{\pi/2} \mathcal{H}_{n,p}(\sin \vartheta, |\mathbf{z}'| \sin \vartheta \cos \varphi) \sin^{n-1} \vartheta \cos^{\frac{(\alpha-1)p+n+1}{p-1}} \vartheta d\vartheta, \end{aligned}$$

where, by (3.10),

$$\mathcal{H}_{n,p}(\sin \vartheta, |\mathbf{z}'| \sin \vartheta \cos \varphi) = \left| ((n+\alpha) \cos^2 \vartheta - \alpha) z_{n+1} + (n+\alpha) |\mathbf{z}'| \cos \vartheta \sin \vartheta \cos \varphi \right|^{p/(p-1)}.$$

Introducing here the parameter $\gamma = |\mathbf{z}'|/z_{n+1}$ and using the equality $|\mathbf{z}'|^2 + z_{n+1}^2 = 1$, we obtain

$$\mathcal{H}_{n,p}(\sin \vartheta, |\mathbf{z}'| \sin \vartheta \cos \varphi) = (1 + \gamma^2)^{-p/2(p-1)} |\mathcal{G}_n(\varphi, \vartheta; \gamma)|^{p/(p-1)}, \tag{3.12}$$

where $\mathcal{G}_n(\varphi, \vartheta; \gamma)$ is given by (3.5).

By (3.9), taking into account (3.11) and (3.12), we arrive at (3.3).

(iii) Let $p = 2$. By (3.3), (3.4) and (3.5),

$$C_2 = \sqrt{\omega_{n-1}} k_{n,\alpha} \sup_{\gamma \geq 0} \frac{1}{\sqrt{1+\gamma^2}} \left\{ \int_0^\pi d\varphi \int_0^{\pi/2} \mathcal{F}_{n,2}(\varphi, \vartheta; \gamma) d\vartheta \right\}^{1/2}, \tag{3.13}$$

where

$$\mathcal{F}_{n,2}(\varphi, \vartheta; \gamma) = \left[((n+\alpha) \cos^2 \vartheta - \alpha) + \gamma(n+\alpha) \cos \vartheta \sin \vartheta \cos \varphi \right]^2 \cos^{n-1+2\alpha} \vartheta \sin^{n-1} \vartheta \sin^{n-2} \varphi.$$

The last equality and (3.13) imply

$$C_2 = \sqrt{\omega_{n-1}} k_{n,\alpha} \sup_{\gamma \geq 0} \frac{1}{\sqrt{1+\gamma^2}} \{ \mathcal{I}_1 + \gamma^2 \mathcal{I}_2 \}^{1/2}, \tag{3.14}$$

where

$$\begin{aligned} \mathcal{I}_1 &= \int_0^\pi \sin^{n-2} \varphi d\varphi \int_0^{\pi/2} ((n+\alpha) \cos^2 \vartheta - \alpha)^2 \sin^{n-1} \vartheta \cos^{n-1+2\alpha} \vartheta d\vartheta \\ &= \frac{\sqrt{\pi} n(n+2)(n+\alpha) \Gamma\left(\frac{n-1}{2}\right) \Gamma\left(\frac{n+2+\alpha}{2}\right)}{4(n+2\alpha)(n+1+\alpha) \Gamma(n+\alpha)} \end{aligned} \tag{3.15}$$

and

$$\begin{aligned}\mathcal{I}_2 &= (n + \alpha)^2 \int_0^\pi \sin^{n-2} \varphi \cos^2 \varphi \, d\varphi \int_0^{\pi/2} \sin^{n+1} \vartheta \cos^{n+1+2\alpha} \vartheta \, d\vartheta \\ &= \frac{\sqrt{\pi} (n + \alpha) \Gamma\left(\frac{n-1}{2}\right) \Gamma\left(\frac{n+2+2\alpha}{2}\right)}{4(n + 1 + \alpha)\Gamma(n + \alpha)}.\end{aligned}\tag{3.16}$$

By (3.14) we have

$$C_2 = \sqrt{\omega_{n-1}} k_{n,\alpha} \max \{\mathcal{I}_1^{1/2}, \mathcal{I}_2^{1/2}\}.\tag{3.17}$$

Further, by (3.15) and (3.16),

$$\frac{\mathcal{I}_1}{\mathcal{I}_2} = \frac{n(n + 2)}{n + 2\alpha}.$$

Therefore,

$$\frac{\mathcal{I}_1}{\mathcal{I}_2} - 1 = \frac{n^2 + n - 2\alpha}{n + 2\alpha}.$$

Taking into account (3.17) and that $n + 2\alpha > 0$ for $p = 2$ by (2.3), we see that inequality

$$\frac{\mathcal{I}_1}{\mathcal{I}_2} \geq 1$$

holds for $\alpha \leq n(n + 1)/2$. So, we arrive at the representation for C_2 with $-(n/2) < \alpha \leq n(n + 1)/2$ given in formulation of the Proposition.

Since $\mathbf{z} \in \mathbb{S}^n$ and the supremum in $\gamma = |\mathbf{z}'|/z_{n+1}$ in (3.13) is attained for $\gamma = 0$, we have $C_2 = C_2(\mathbf{e}_{n+1})$ under requirements of the Proposition. \square

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